Efficient parallel string comparison

Peter Krusche and Alexander Tiskin

Department of Computer Science

THE UNIVERSITY OF WARWICK

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Outline

Bulk-Synchronous Parallelism

Model for parallel computation:

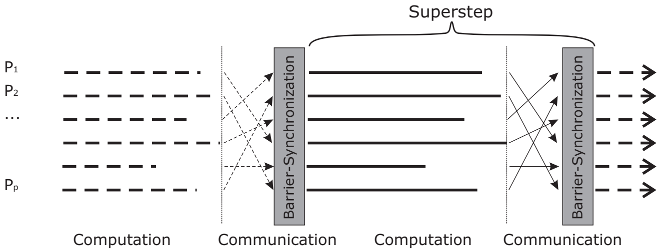
L. G. Valiant.
A bridging model for parallel computation.
Communications of the ACM, 33:103–111, 1990.

Main ideas:

- p processors working asynchronously
- Can communicate using g operations to transmit one element of data
- Can synchronise using l sequential operations.

Bulk-Synchronous Parallelism

- Computation proceeds in *supersteps*
- Communication takes place at the end of each superstep
- Between supersteps, barrier-style synchronisation takes place



Bulk-Synchronous Parallelism

Superstep s has computation cost w_s and communication $h_s = \max(h_s^{in}, h_s^{out})$.

When there are S supersteps:

$\Rightarrow$ Computation work

$$W = \sum_{1 \leqslant s \leqslant S} w_s$$

$\Rightarrow$ Communication

$$H = \sum_{1 \leqslant s \leqslant S} h_s$$

Formula for running time: $T = W + g \cdot H + l \cdot S$.

Definition (Parallel Prefix)

Given n values $x_1, x_2, \ldots, x_n$ and an associative operator $\oplus$, compute the values x_1, $x_1 \oplus x_2$, $x_1 \oplus x_2 \oplus x_3$, $\ldots$, $\bigoplus_{i=1,2,\ldots,n} x_i$.

Fact...

Under some natural assumptions, we can carry out a parallel prefix operation over n elements on a BSP computer with p processors using $W = O(\frac{n}{p})$, $H = O(p)$ and $S = O(1)$.

Definition (Input data)

Let $x = x_1x_2 \ldots x_m$ and $y = y_1y_2 \ldots y_n$ be two strings on an alphabet Σ.

Definition (Subsequences)

A *subsequence* u of x: u can be obtained by deleting zero or more elements from x.

Definition (Longest Common Subsequences)

An *LCS* (x, y) is any string which is subsequence of both x and y *and* has maximum possible length. Length of these sequences: *LLCS* (x, y).

The Semi-local LCS Problem

Definition (Substrings)

A *substring* of any string x can be obtained by removing zero or more characters from the beginning and/or the end of x.

Definition (Highest-score matrix)

The element $A(i, j)$ of the LCS *highest-score matrix* of two strings x and y gives the LLCS of $y_i \ldots y_j$ and x.

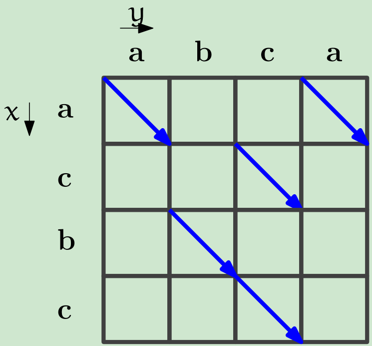
Definition (Semi-local LCS)

Solutions to the semi-local LCS problem are represented by a (possibly implicit) highest-score matrix $A(i, j)$.

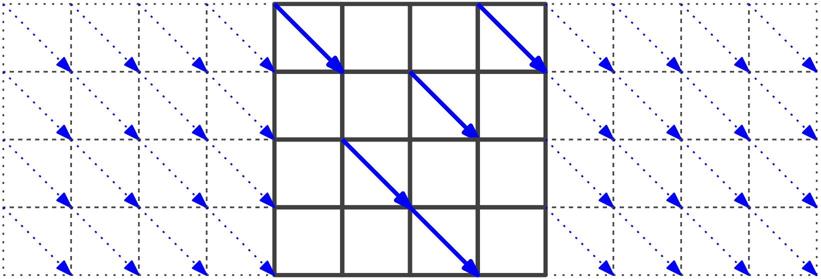
LCS grid dags and highest-score matrices

- LCS Problem can be represented as longest path problem in a Grid DAG
- String-Substring LCS Problem $\Rightarrow$ $A(i,j) =$ length of longest path from $(0,i)$ to (n,j) (top to bottom).

Example



Infinite extension of the LCS grid dag, outside the core area, everything matches:



The extended highest-score matrix is now defined on indices $[-\infty, +\infty] \times [-\infty, +\infty]$.

Additional definitions

Definition (Integer ranges)

We denote the set of integers $\{i, i+1, \ldots, j\}$ as $[i : j]$.

Definition (Odd half-integers)

We denote half-integer variables using a $\hat{}$, and denote the set of half-integers $\{i + \frac{1}{2}, i + \frac{3}{2}, \ldots, j - \frac{1}{2}\}$ as $\langle i : j \rangle$.

Definition (Critical Point)

Odd half-integer point $(i-\frac{1}{2}, j+\frac{1}{2})$ is *critical* iff.
$A(i,j)+1 = A(i-1,j) = A(i,j+1) = A(i-1,j+1)$.

Theorem (Schmidt'95, Alves+'06, Tiskin'05)

1. *We can represent a the whole extended highest-score matrix by a finite set of such critical points.*
2. *Assuming w.l.o.g. input strings of equal length* n, *there are* $N = 2n$ *such critical points that implicitly represent the whole score matrix.*
3. *There is an algorithm to obtain these points in time* $O(n^2)$.

Example (Explicit highest-score matrix)

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**
-4	4	4	4	4	4	4	4	4
-3	3	3	3	4	4	4	4	4
-2	2	2	3	4	4	4	4	4
-1	1	1	2	3	3	3	4	4
0	0	1	2	3	3	3	4	4
1	-1	0	1	2	2	2	3	4
2	-2	-1	0	1	1	2	3	4
3	-3	-2	-1	0	1	2	3	4

Querying highest-score matrix entries

Theorem (Tiskin'05)

If $d(i,j)$ *is the number of critical points* $(\hat{\imath},\hat{\jmath})$ *in the extended score matrix with* $i < \hat{\imath}$ *and* $\hat{\jmath} < j$, *then* $A(i,j) = j - i - d(i,j)$.

Definition (Density and distribution matrices)

The elements $d(i,j)$ form a *distribution matrix* over the entries of density (permutation) matrix D with nonzeros at all critical points $(\hat{\imath},\hat{\jmath})$ in the extended highest-score matrix:

$$d(i,j) = \sum_{(\hat{\imath},\hat{\jmath}) \in \langle i:N \rangle \times \langle 0:j \rangle} D(\hat{\imath},\hat{\jmath})$$

Can compute this efficiently using range querying data structures.

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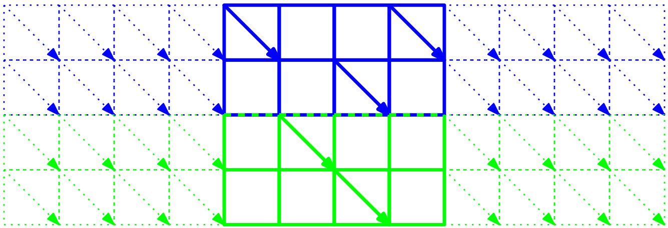
Outline

Sequential highest-score matrix multiplication

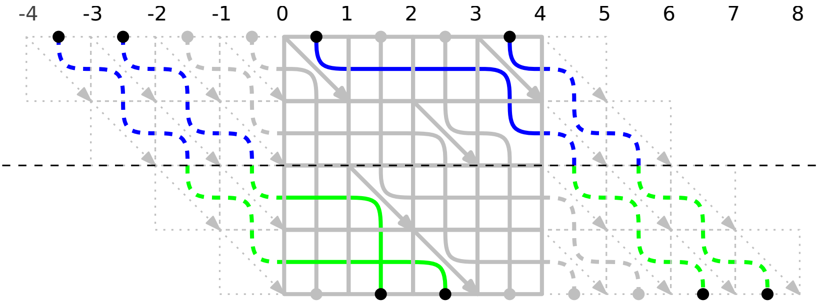
Algorithm, Tiskin'05

Given the distribution matrices d_A and d_B for two adjacent blocks of equal height M and width N in the grid dag, we can compute the distribution matrix d_C for the union of these blocks in $O(N^{1.5} + M)$.



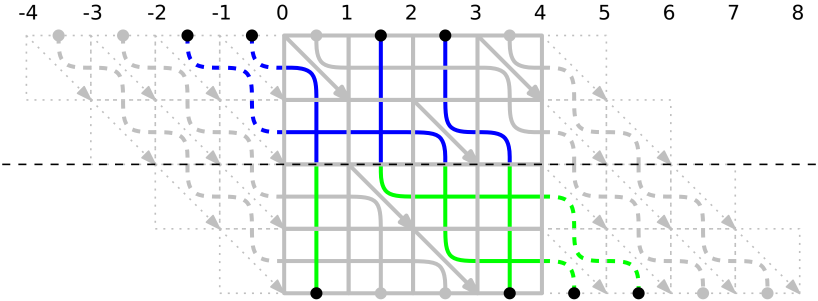
Combine critical points by removing double crossings

“Trivial part” ($O(M)$):



Combine critical points by removing double crossings

“Nontrivial part” ($O(N^{1.5})$):



Nontrivial part

- The non-trivial part can be seen as $(\min, +)$ matrix product ($d_{A|B|C}$ are now the nontrivial parts of the corresponding distribution matrices):

$$d_C(i,k) = \min_j (d_A(i,j) + d_B(j,k))$$

- Explicit form, naive algorithm: $O(n^3)$
- Explicit form, algorithm that uses highest-score matrix properties: $O(n^2)$
- Implicit form, divide-and-conquer: $O(n^{1.5})$

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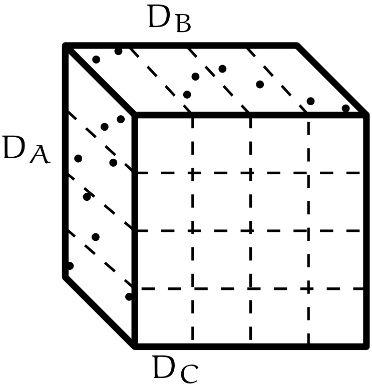
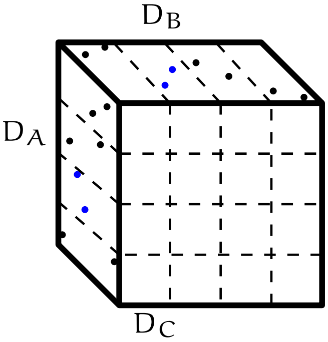
Divide-and-conquer multiplication

C-blocks and relevant nonzeros

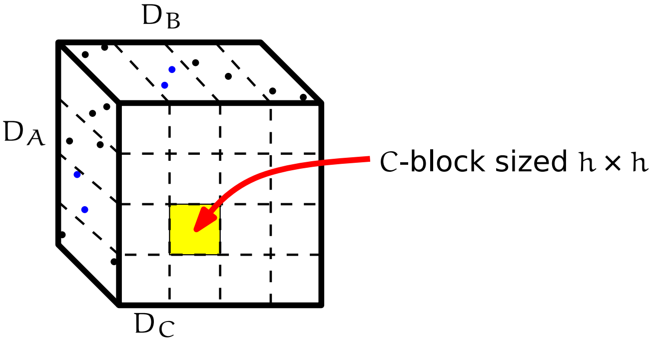


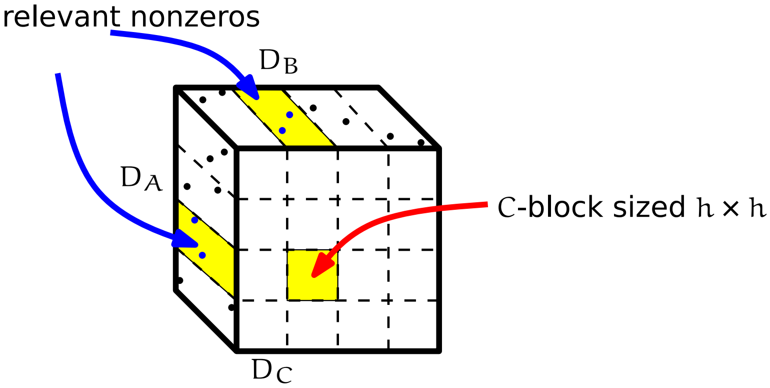
relevant nonzeros



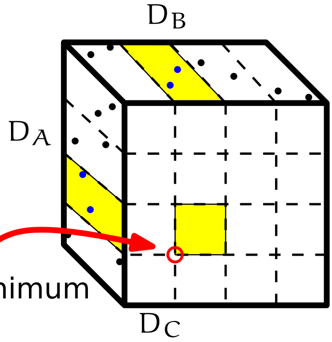
C-block sized $h \times h$

relevant nonzeros



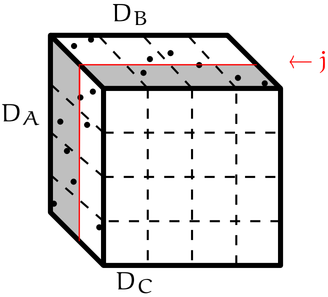

relevant nonzeros
D_B
D_A
C-block sized $h \times h$
D_C

relevant nonzeros



local minimum

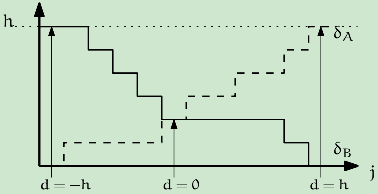
C-block sized $h \times h$



Splitting relevant nonzeros in D_A and D_B into two sets at a position $j \in [0 : N]$, we get numbers

- $\delta_A^{\square}(j)$ of relevant nonzeros in D_A up to column $j - \frac{1}{2}$
- $\delta_B^{\square}(j)$ of relevant nonzeros in D_B starting at row $j + \frac{1}{2}$

Example ($\jmath$-blocks)



Definition (Δ-sequences)

δ's don't change inside a $\jmath$-block $\Rightarrow$

$$\Delta_A^{\square}(d) = \operatorname*{any}_{j \in \mathcal{J}^{\square}(d)} \delta_B^{\square}(j)$$

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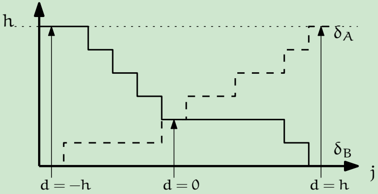
Example (j-blocks)



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Definition (local minima)

The sequence

$$M^{\square}(d) = \min_{j \in \mathcal{J}^{\square}(d)} \left(d_A(i_0, j) + d_B(j, k_0)\right)$$

contains the minimum of $d_A(i_0, j) + d_B(j, k_0)$ in every j-block.

We can use M's and Δ's to compute the number of nonzeros in a C-block.

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We can use M's and Δ's to compute the number of nonzeros in a C-block.

Sequences M for every C-subblock can be computed in $O(h)$:

$$
\begin{aligned}
M^{\text{◱}}(d') &= \min_{d} M^{\square}(d), \\
M^{\text{◲}}(d') &= \min_{d} M^{\square}(d) + \bar{\Delta}_B^{\text{◲}}(d), \\
M^{\text{◰}}(d') &= \min_{d} M^{\square}(d) + \bar{\Delta}_A^{\text{◰}}(d), \\
M^{\text{◳}}(d') &= \min_{d} M^{\square}(d) + \bar{\Delta}_A^{\text{◳}}(d) + \bar{\Delta}_B^{\text{◳}}(d)
\end{aligned}
$$

having $\bar{\Delta}_A^{(i',k',\frac{h}{2})}(d) - \bar{\Delta}_B^{(i',k',\frac{h}{2})}(d) = d'$ with $d' \in [-\frac{h}{2} : \frac{h}{2}]$.

Divide-and-conquer multiplication
Recursive step

- Sequences $\Delta_A(d')$ and $\Delta_B(d')$ can also be determined in $O(h)$ by a scan of the relevant nonzeros for each subblock.
- Knowing $\Delta_A(d')$, $\Delta_B(d')$ and $M(d')$ for each subblock, we can continue the recursion in every subblock.
- The recursion terminates when N C-blocks of size 1 are left.

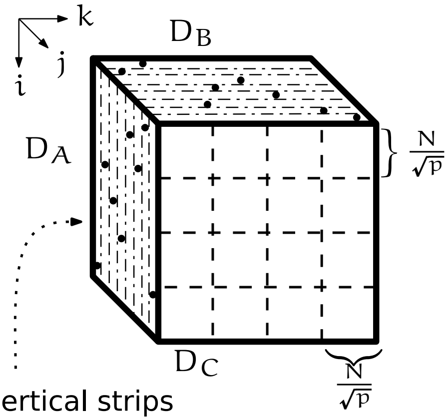
Outline

- Start the recursion at a point where there are p C-blocks.
- This is at level $\frac{1}{2}\log p$.
- Precompute and distribute the required sequences Δ and M for each C-block in parallel.
- Every C-block has size $h = \frac{N}{\sqrt{p}}$, and hence requires sequences with $O(\frac{N}{\sqrt{p}})$ values.
- After these sequences have been precomputed and redistributed, we can use the sequential algorithm to finish the computation.

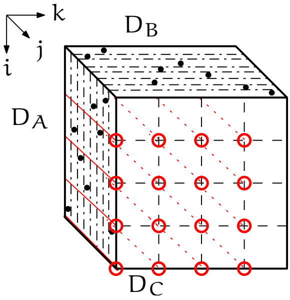
Assume that:

- $\sqrt{p}$ is an integer.
- Every processor has unique identifier q with $0 \leqslant q < p$.
- Every processor q corresponds to exactly one location $(q_x, q_y) \in [0 : \sqrt{p} - 1] \times [0 : \sqrt{p} - 1]$.
- Initial distribution of nonzeros in D_A and D_B is assumed to be even among all processors.

- Redistribute the nonzeros to strips of width $\frac{N}{p}$
- Send all nonzeros $(\hat{\imath}, \hat{\jmath})$ in D_A and $(\hat{\jmath}, \hat{k})$ in D_B to processor $\lfloor (\hat{\jmath} - \frac{1}{2}) \cdot p/N \rfloor$.
- Possible in one superstep using communication $O(\frac{N}{p})$.

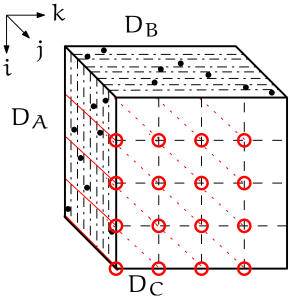


p vertical strips
$\frac{N}{p}$ nonzeros each



1. Compute the elementary $(\min, +)$ products $d_A(o_x, j) + d_B(j, o_y)$ along $j \in [0 : N]$.
2. Processor q holds all $D_A(\hat{\imath}, \hat{\jmath})$ and all $D_B(\hat{\jmath}, \hat{k})$ for $\hat{\jmath} \in \langle q \cdot \frac{N}{p} : (q+1) \cdot \frac{N}{p} \rangle$.
3. Can compute the values $d_A(o_x, j)$ and $d_B(j, o_y)$ by using parallel prefix/suffix.
4. After prefix and suffix computations, every processor holds N/p values $d_A(o_x, j) + d_B(j, o_y)$ for $j \in [q \cdot \frac{N}{\sqrt{p}} : (q+1) \cdot \frac{N}{\sqrt{p}}]$.

Precomputing M



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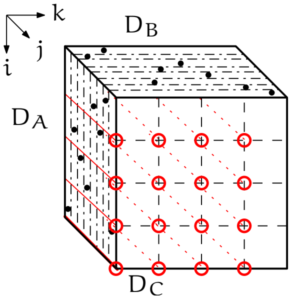

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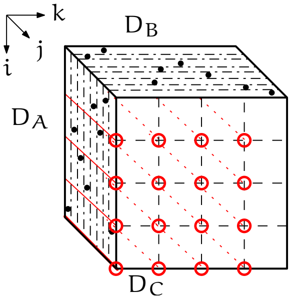

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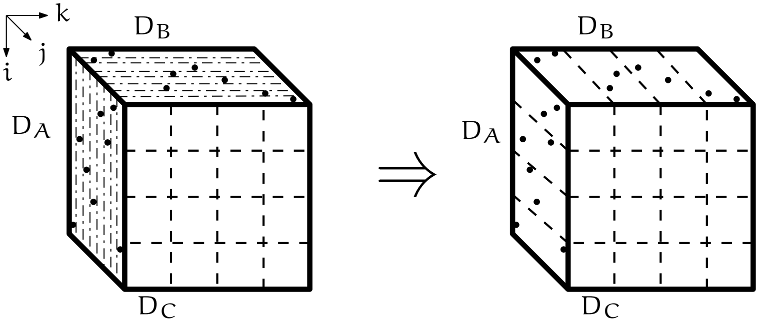
Redistribution Step



p vertical strips $\frac{N}{p}$ nonzeros each

$\sqrt{p}$ horizontal strips with $\frac{N}{\sqrt{p}}$ nonzeros each

- Computational work bounded by the sequential recursion:
  $W = O((N/\sqrt{p})^{1.5}) = O(N^{1.5}/p^{0.75})$
- Every processor holds $O(N/p)$ nonzeros before redistribution.
- Every nonzero is relevant for $\sqrt{p}$ C-blocks.

$\Rightarrow$ $O(N/\sqrt{p})$ communication for redistributing the nonzeros.

$\Rightarrow$ $H = O(N/\sqrt{p} + p + N/\sqrt{p}) = O(N/\sqrt{p})$
(if $N/\sqrt{p} > p \quad \rightarrow N > p^{1.5}$)

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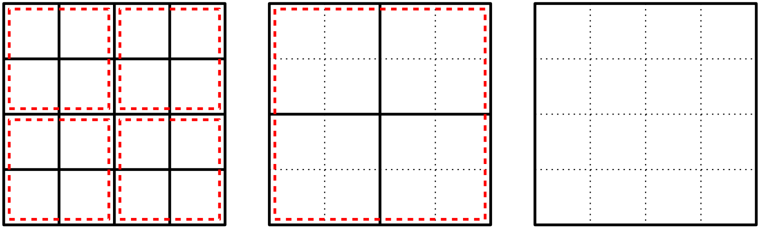
1. Introduction
   - Bulk-Synchronous Parallelism
   - String comparison
2. Sequential LCS algorithms
   - Sequential semi-local LCS
   - Divide-and-conquer semi-local LCS
3. The parallel algorithm
   - Parallel score-matrix multiplication
   - Parallel LCS computation

- First, compute scores for a regular grid of p sub-dags of size $n/\sqrt{p} \times n/\sqrt{p}$
- Then merge these in a quadtree-like scheme using parallel score-matrix multiplication:



Analysis

- Quadtree has $\frac{1}{2}\log_2 p$ levels
- On level l, $0 \leqslant l \leqslant \frac{1}{2}\log_2 p$, we have
  - $p_l = \frac{p}{4^l}$ (number of processors that work together on one merge)
  - $N_l = \frac{N}{2^l}$ (block size of merge)

$$\Rightarrow \; w_l = O\left(\frac{\left(\frac{N}{2^l}\right)^{1.5}}{\left(\frac{p}{4^l}\right)^{0.75}}\right) = O\left(\frac{N^{1.5}}{p^{0.75}}\right)$$

$$\Rightarrow \; h_l = O\left(\frac{\frac{N}{2^l}}{\left(\frac{p}{4^l}\right)^{0.5}}\right) = O\left(\frac{N}{p^{0.5}}\right)$$

- Quadtree has $\frac{1}{2}\log_2 p$ levels

  Hence, we get
- work $W = O(\frac{n^2}{p} + \frac{N^{1.5}\log p}{p^{0.75}}) = O(n^2/p)$ (assuming that $n \geqslant p^2$),
- communication $H = O(\frac{n \log p}{\sqrt{p}})$, and
- $S = O(\log p)$ supersteps.

Comparison to other parallel algorithms

W	H	S	**References**
	Global LCS		
$O(\frac{n^2}{p})$	$O(n)$	$O(p)$	[McColl'95]+ [Wagner & Fischer'74]
	String-Substring LCS		
$O(\frac{n^2}{p})$	$O(Cp^{1/C} n \log p)$	$O(\log p)$	[Alves+'03]
$O(\frac{n^2}{p})$	$O(n \log p)$	$O(\log p)$	[Tiskin'05], [Alves+:06]
	String-Substring, Prefix-Suffix LCS		
$O(\frac{n^2 \log n}{p})$	$O(\frac{n^2 \log p}{p})$	$O(\log p)$	[Alves+'02]
$O(\frac{n^2}{p})$	$O(n)$	$O(p)$	[McColl'95]+ [Alves+'06], [Tiskin'05]
$O(\frac{n^2}{p})$	$O(\frac{n \log p}{\sqrt{p}})$	$O(\log p)$	NEW

Summary

We have looked at a parallel algorithm for semilocal string comparison that is

- communication efficient

  (in fact, achieving scalable communication),
- work-optimal,

! and asymptotically better than even global LCS computation.

Outlook

- Score matrix multiplication can also be applied to create a scalable algorithm for the longest increasing subsequence problem.
- Algorithm can be adapted to compute edit-distances.

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Thank you!

Any questions?